

HARMONIC ANALYSIS AND MEASURE PARTITIONS I: GRÜNBAUM AND MAKEEV PROBLEMS FOR COMPLEX REGULAR FANS

STEVEN SIMON

ABSTRACT. Group-theoretic generalizations of the Ham Sandwich Theorem, seen as the “balancing” of finite measures by linear representations, are shown for abelian groups to have an equivalent interpretation in terms of vanishing Fourier transforms. We consider the finite cases, which yield measure partitions by collections F_{q_1}, \dots, F_{q_k} of complex regular q_j -fans, analogues of the famous Grünbaum problem on equipartitions by families of hyperplanes (i.e., regular 2-fans). In addition to equipartitions by k -tuples of complex regular p -fans for odd primes p , the Fourier method allows for cohomological techniques in \mathbb{Z}_q for non-prime q , thereby producing equipartitions by a regular 9-fan as well as “modulo” equipartitions by non prime-power numbers of regions: e.g., if $q_j = pr_j$, the existence of k complex regular q_j -fans, each of whose sub-families of k -tuples of complex regular p -fans equipartitions each measure.

1. INTRODUCTION

1.1. Balancing Measures by Linear Representations. Mass partition problems occupy a central position in geometric combinatorics, with the most classical result being the Ham Sandwich Theorem – any d measures on \mathbb{R}^d can be simultaneously bisected by a single hyperplane. This paper provides a continuation of the group theoretic generalizations of this theorem, first discussed in [21], which aim to realize group representations as corresponding group symmetries of finite measures on Euclidian space.

In each case, a Lie group G gives rise to a canonical class of decompositions $\mathbb{R}^d = \cup_{g \in G} \mathcal{R}_g$ by convex domains $\{\mathcal{R}_g\}_{g \in G}$, naturally associated to G in that all G -decompositions can be realized by the base manifold $M_d(G)$ of a principle G -bundle $G \hookrightarrow E_d(G) \rightarrow M_d(G)$, with the proper and transitive G -action on each decomposition realized by the free G -action on each fiber of $E_d(G)$, and moreover that the classifying bundle $G \hookrightarrow EG \rightarrow BG$ for G (see, e.g. [12]) is the limit of this construction as $d \rightarrow \infty$.

Given a m -tuple $\mu = (\mu_1, \dots, \mu_m)$ of mass distributions on \mathbb{R}^d and any finite representation $\rho : G \rightarrow O(n)$, one can consider the “ (ρ, G) -average” of the measures of the \mathcal{R}_g of any G -decomposition, where this average takes the form

$$(1.1) \quad \sum_{g \in G} \rho_g^{-1}(\mu(\mathcal{R}_g)) = \sum_{g \in G} g^{-1} \cdot \mu(g \cdot \mathcal{R}_e) \in \mathbb{R}^n$$

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in the finite cases and

$$(1.2) \quad \int_G \rho_g^{-1}(\mu(\mathcal{R}_g)) dg = \int_G g^{-1} \cdot \mu(g \cdot \mathcal{R}_e) dg \in \mathbb{R}^n$$

when the group is continuous, the integral being taken with respect to normalized Haar measure.

In a general sense, the G -average (1.1) or (1.2) evaluates the symmetry of the measures of the regions of a G -decomposition with respect to the given representation, and we therefore say that a decomposition “ $G = (\rho, G)$ -balances” a m -tuple of measures if this average is zero. By a “ (ρ, G) -Ham Sandwich Theorem,” we mean a result which prescribes conditions under which any m -tuple of measures on \mathbb{R}^d can be (ρ, G) -balanced by some G -decomposition. Such theorems can be seen as “dual” to the configuration-space/test-map paradigm, ubiquitous throughout topological combinatorics, whereby a given problem of combinatorial or discrete geometry is reduced to an equivariant topological framework to which the robust machinery of algebraic topology – e.g., cohomological index theory, (equivariant) characteristic and other obstruction classes, spectral sequences, etc. – can be applied. Surveys of these methods can be found in [16], [31], [32], and [33], among others. By contrast, we begin with group actions, the topology of which is used to prove naturally associated families of measure partitions.

1.2. A Harmonic Analysis Interpretation. The main observation of the present paper is that for abelian groups and unitary representations, there is an equivalent formulation of G -Ham Sandwich Theorems in terms of harmonic analysis. We deal with the finite cases, in which case each $\{\mathcal{R}_g\}_{g \in G}$ forms a partition of \mathbb{R}^d . A discussion for continuous Lie Groups is given in [22].

Given any m measures μ_1, \dots, μ_m , each G -partition $\{\mathcal{R}_g\}_{g \in G}$ determines m mappings $f_i : G \rightarrow \mathbb{R}$, $f_i(g) = \mu_i(\mathcal{R}_g)$, $1 \leq i \leq m$. Since G is abelian, its irreducible unitary representations $\chi : G \rightarrow U(1)$ are all 1-dimensional, so that the Fourier expansion of f_i takes the simple form

$$(1.3) \quad \mu_i(\mathcal{R}_g) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} c_i^\chi \chi(g),$$

where \hat{G} is the Pontryagin dual of G consisting of all non-isomorphic irreducible unitary representations of G , naturally identified with G itself (see, e.g., [20] and [24]). The Fourier coefficients (transforms)

$$(1.4) \quad c_i^\chi = \sum_{g \in G} \mu_i(\mathcal{R}_g) \chi(g^{-1})$$

of the f_i are thus the (χ, G) -averages of the μ_i , and as any unitary $\rho = \oplus_{r=1}^n \chi_r$ is a direct sum of $\chi_r \in \hat{G}$, it follows that any (ρ, G) -Ham Sandwich Theorem is the simultaneous vanishing of prescribed coefficients. In particular, the coefficient $c_i^{\chi_0} = \sum_{g \in G} \mu_i(\mathcal{R}_g)$ of the trivial representation is just $\mu(\mathbb{R}^d)$, so that equipartitions in the usual sense – $\mu_i(\mathcal{R}_g) = \mu_i(\mathbb{R}^d)/|G|$ for each $1 \leq i \leq m$ and each $g \in G$ – are those (ρ, G) -Ham Sandwich Theorems for which all other coefficients vanish.

As is often the case in topological combinatorics (see, e.g., [2], [5], [15], [33]), the central result Theorem 3.1 is stated as a polynomial condition arising from cohomology rings, here a relevant Chern class. For the elementary abelian groups $G = Z_p^k$, p an odd prime, one is reduced to calculations in the field \mathbb{Z}_p as usual, thereby yielding equipartitions of \mathbb{C}^d by the p^k regions determined by k complex regular p -fans in general position (Theorem 4.3 and Corollaries 4.4 and 4.5).

For general abelian groups and other representations, however, the Fourier method allows one to carefully select the coefficients to be annihilated, thereby affording some computation with zero divisors in \mathbb{Z}_q when q is non-prime. This allows for some measure partitions unobtainable by ordinary cohomological methods, such as the equipartition of a single measure by a complex regular 9-fan (Proposition 4.2). Additionally, one has a family of “modulo” equipartitions for complex regular fans when the number of regions is not a prime power (Proposition 5.1, Theorem 5.2, and Corollary 5.3.). These stand in partial contrast to previous partition theorems obtained by equivariant methods, which apply to collections of regions (or, in the case of the topological or colored Tverberg problem, faces) of prime or prime-power order (see, e.g., [1], [2], [3], [4], [13], [15], [19], [25], [27], etc.). We also indicate how combining multiple Fourier partitions for a given G , when coupled with a possible manifold intersection argument, should produce *full* equipartitions by an arbitrary number of regions. Finally, we believe similar harmonic methods will apply to other problems of topological combinatorics, and to the non-prime-power cases of the topological Tverberg problem especially [23].

2. THE GRÜNBAUM PROBLEM AND Z_2^k -HAM SANDWICHES

First, we show how perhaps the most famous problem of equipartition theory, the Grünbaum problem, can be seen as a special $Z_2^k = \{\pm 1\}^k$ -case of our procedure.

The contemporary reformulation of the Grünbaum problem [8] asks for the minimum dimension $d = \Delta(m, k)$ for which any any m finite, signed proper Borel measures on \mathbb{R}^d – those for which each hyperplane is a null set and for which \mathbb{R}^d has non-zero measure, henceforth to be called simply a *measure* – can be equipartitioned into 2^k orthants by k hyperplanes in general position. This problem is extensively studied (see, e.g., [2], [5], [10], [15], [18], [30]) with the greatest number of values and estimates given so far in [15], [18], and [30].

To rephrase the problem in terms of Z_2^k -averages, first consider the standard Z_2^k -action on $E_d(Z_2^k) = (S^d)^k$. Each $x_j = (\mathbf{a}_j, b_j) \in S^d$, $\|\mathbf{a}_j\|^2 + |b_j|^2 = 1$, determines a hyperplane $H_j = \{\mathbf{u} \in \mathbb{R}^d \mid \langle \mathbf{u}, \mathbf{a}_j \rangle = b_j\}$ if $\mathbf{a}_j \neq 0$ and a “hyperplane at infinity” otherwise, so each Z_2^k -orbit $\{\delta \cdot x\}_{\delta \in Z_2^k}$ of $x = (x_1, \dots, x_k) \in E_d(Z_2^k)$ determines k hyperplanes (some possibly at infinity) and a corresponding partition of \mathbb{R}^d into (not-necessarily distinct) regions $\mathcal{O}_\delta = \{\mathbf{u} \in \mathbb{R}^d \mid (\forall 1 \leq j \leq k) (\exists v_j \geq 0) \langle \mathbf{u}, \mathbf{a}_j \rangle - b_j = \delta_j v_j\}$, $\delta = (\delta_1, \dots, \delta_k)$ (see, e.g., [2]). Ignoring the symmetric group action on the k coordinates of $E_d(Z_2^k)$ (though see [5] and especially [30] for the use of the full Dihedral D_8 -action when $k = 2$), this association is unique. The proper and transitive Z_2^k -action $\delta_1 \cdot \mathcal{O}_{\delta_2} := \mathcal{O}_{\delta_1 \delta_2}$ on each decomposition (generated by reflections about each

of k hyperplanes in general position in the case of generic $x \in E_d(S^k)$ corresponds precisely to the free Z_2^k -action on $E_d(Z_2^k)$, so that each Z_2^k -partition is some point of the k -fold product $(\mathbb{R}P^{d-1})^k$ of real projective space.

On the other hand, the non-isomorphic irreducible representations of Z_2^k are all real, indexed by $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{Z}_2^k$ and given explicitly by $\chi^\epsilon : Z_2^k \rightarrow O(1)$, $\chi^\epsilon(\delta) = \prod_{j=1}^k \delta_j^{\epsilon_j}$. For a fixed decomposition $\{\mathcal{O}_\delta\}_{\delta \in Z_2^k}$ and any m measures μ_1, \dots, μ_m on \mathbb{R}^d , the Fourier coefficients $c_i^\epsilon = \sum_{\delta \in Z_2^k} f_i(\delta) \chi^\epsilon(\delta^{-1})$ of $f_i(\delta) = \mu_i(\mathcal{O}_\delta) = 2^{-k} \sum_{\epsilon} c_i^\epsilon \chi^\epsilon(\delta)$ are just the (Z_2^k, χ^ϵ) -averages of the μ_i , and $c_i^0 = \mu_i(\mathbb{R}^d)$. Thus if $\rho = m \oplus_{\epsilon \neq 0} \chi^\epsilon$ (equivalent to the use, as in [15], of the regular representation $\mathbb{R}[Z_2^k]$ in the configuration-space/test-map procedure), and $\mu = (\mu_1, \dots, \mu_1, \dots, \mu_m, \dots, \mu_m)$, the vanishing of the (ρ, Z_2^k) -average $\sum_{\delta} \rho_\delta^{-1}(\mu(\mathcal{O}_\delta)) \in \mathbb{R}^{(2^k-1)m}$ is the equipartition of each measure by k (necessarily distinct) hyperplanes.

3. HAM SANDWICH THEOREMS FOR FINITE ABELIAN GROUPS

More generally, each finite abelian group can be expressed as a product of cyclic groups $G = Z_{q_1} \times \dots \times Z_{q_k}$, with each $Z_s = \{\zeta_s^k\}_{k=0}^{s-1}$ identified with the s -th roots of unity, $\zeta_s = e^{2\pi i/s}$. The standard G -action on $E_d(G) = [S(\mathbb{C}^{d+1})]^k$ is given by $\zeta \cdot (x_1, \dots, x_k) = (\zeta_{q_1}^{j_1} x_1, \dots, \zeta_{q_k}^{j_k} x_k)$. As before, we write each $x_j \in S(\mathbb{C}^{d+1})$ as $x_j = (\mathbf{a}_j, b_j)$, $\|\mathbf{a}_j\|^2 + |b_j|^2 = 1$. Let $R_r(s) = \{v \in \mathbb{C} \mid \arg(v) \in [(r-1)/s, (r+1)/s]\}$ denote the regular s -sectors of \mathbb{C} centered at the origin, $0 \leq r < s$. Provided $\mathbf{a}_j \neq 0$, the Z_{q_j} -orbit $\{\zeta_{q_j}^{r_j} x_j\}_{j=0}^{q_j-1}$ of each x_j determines a partition of \mathbb{C}^d into the regular sectors

$$(3.1) \quad \mathcal{S}_{r_j}(q_j) = \{\mathbf{u} \in \mathbb{C}^d \mid (\exists v_j \in R_{r_j}(q_j)) \langle \mathbf{u}, \mathbf{a}_j \rangle_{\mathbb{C}} - \bar{b}_j = v_j\}$$

of the regular q_j -fan

$$(3.2) \quad F_{q_j} = \{\mathbf{u} \in \mathbb{C}^d \mid (\exists 0 \leq r < q_j) \langle \mathbf{u}, \mathbf{a}_j \rangle_{\mathbb{C}} - \bar{b}_j = \zeta_{q_j}^r\}$$

centered about the complex hyperplane

$$(3.3) \quad H_j^{\mathbb{C}} = \{\mathbf{u} \in \mathbb{C}^d \mid \langle \mathbf{u}, \mathbf{a}_j \rangle_{\mathbb{C}} = \bar{b}_j\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{C}} = \sum_{i=1}^d u_i \bar{v}_i$ is the standard Hermitian form on \mathbb{C}^d . If $\mathbf{a}_j = 0$, we say that the F_{q_j} given by (3.2) is centered “at infinity”, with $\mathcal{S}_{r_j}(q_j)$ given by (3.1) empty or all of \mathbb{C}^d . Each G -orbit $\{\zeta \cdot x\}_{\zeta \in G}$ therefore determines a partition $\{\mathcal{R}_\zeta\}_{\zeta \in G}$ of \mathbb{C}^d , where each (not necessarily distinct) $\mathcal{R}_\zeta = \cap_{j=1}^k \mathcal{S}_{r_j}(q_j)$ is the intersection of the $\mathcal{S}_{r_j}(q_j)$. We call such a partition *non-trivial* if at least one of the F_{q_j} is not centered at infinity. In particular, each partition of \mathbb{C}^d into $Q = \prod_{j=1}^k q_j$ distinct regions determined by k complex regular q_j -fans is realized by the generic $x \in E_d(G)$. Elementary properties of complex conjugation show that the proper and transitive G -action $\zeta_1 \cdot \mathcal{R}_{\zeta_2} := \mathcal{R}_{\zeta_1 \zeta_2}$ on $\{\mathcal{R}(\zeta)\}_{\zeta \in G}$ (generated by rotations by multiples of $2\pi/q_j$ about each complex hyperplane $H_j^{\mathbb{C}}$ in the generic cases) corresponds precisely to the G -action on the orbit $\{\zeta \cdot x\}_{\zeta \in G} \subset E_d(G)$, so one has a natural identification of each decomposition with a point of the product $M_d(G) = \prod_{j=1}^k L^{2d+1}(q_j)$ of standard lens spaces $L^{2d+1}(q_j) = S(\mathbb{C}^{d+1})/Z_{q_j}$.

The irreducible unitary representations of G are natural indexed by $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \bigoplus_{j=1}^k \mathbb{Z}_{q_j}$ and given explicitly by $\chi^\epsilon(\zeta) = \prod_{j=1}^k \zeta^{\epsilon_j}$, with any n -dimensional unitary representation $\rho : G \rightarrow U(n)$ a direct sum of the χ^ϵ . As each representation is unitary, we consider n -tuples $\mu = (\mu_1, \dots, \mu_n)$ of finite *complex*-valued measures on \mathbb{C}^d (thus each μ_i is a pair of real-valued measures) and the (ρ, G) -averages $\sum_{\zeta \in G} \rho_\zeta^{-1}(\mu(\mathcal{R}_\zeta)) \in \mathbb{C}^n$. One then has the following theorem:

Theorem 3.1 (Ham Sandwich Theorem for Finite Abelian Groups). *Let $\rho = \bigoplus_{r=1}^n \chi^{\epsilon_r} : G \rightarrow U(n)$ be a unitary representation of $G = \prod_{j=1}^k \mathbb{Z}_{q_j}$, $\epsilon_r = (\epsilon_{r,1}, \dots, \epsilon_{r,k}) \in \bigoplus_{j=1}^k \mathbb{Z}_{q_j}$. If*

$$(3.4) \quad f(y_1, \dots, y_k) = \prod_{r=1}^n (\epsilon_{r,1} y_1 + \dots + \epsilon_{r,k} y_k) \in \mathbb{Z}[y_1, \dots, y_k] / \mathcal{I}$$

is non-zero, $\mathcal{I} = \langle q_j y_j, y_j^{d+1} \mid 1 \leq j \leq k \rangle$, then any n -tuple $\mu = (\mu_1, \dots, \mu_n)$ of complex-valued measures on \mathbb{C}^d can be (ρ, G) -balanced by a non-trivial G -partition $\{\mathcal{R}_\zeta\}_{\zeta \in G}$:

$$(3.5) \quad \sum_{\zeta \in G} \rho_\zeta^{-1}(\mu(\mathcal{R}_\zeta)) = 0$$

The proof of Theorem 3.1, which we defer to section 6, rests ultimately on the non-vanishing of the top Chern class of a complex vector bundle associated to the representation ρ , thereby creating the polynomial condition (3.4). For now, we present applications of Theorem 3.1 to real-valued measures.

As before, each partition $\{\mathcal{R}_\zeta\}_{\zeta \in G}$ yields m mappings $f_i : G \rightarrow \mathbb{C}$,

$$(3.6) \quad f_i(\zeta) = \mu_i(\mathcal{R}_\zeta) = \frac{1}{|G|} \sum_{\epsilon \in \bigoplus_{j=1}^k \mathbb{Z}_{q_j}} c_i^\epsilon \chi^\epsilon(\zeta)$$

whose Fourier coefficients are the (χ^ϵ, G) -averages of μ_i . Thus the (ρ, G) -Ham Sandwich Theorem is equivalent to the vanishing of the prescribed Fourier coefficients $c_i^{\epsilon_r}$. In particular, $c_i^{-\epsilon} = \overline{c_i^\epsilon} = 0$ when the measures are real-valued, and again $c_i^0 = \mu_i(\mathbb{C}^d)$. Thus the vanishing of each c_i^ϵ with $\epsilon \in [\Pi] := \{\epsilon \neq 0 \mid 0 \leq \epsilon_k \leq \lceil (q_k - 1)/2 \rceil\}$ is equivalent to the equipartition of each measure by (necessarily distinct) complex regular fans F_{q_1}, \dots, F_{q_k} , i.e., to the (ρ, G) -balancing of $\mu = (\mu_1, \dots, \mu_1, \dots, \mu_m, \dots, \mu_m)$ by the representation $\rho = m \bigoplus_{\epsilon \in [\Pi]} \chi^\epsilon$.

As a real hyperplane is a regular 2-fan, the problem of equipartitioning any m measures by complex regular fans F_{q_1}, \dots, F_{q_k} is a complex analogue of the classical Grünbaum problem above. We therefore pose the following:

Question 1. *What is the smallest dimension $d = \Delta_{\mathbb{C}}(m; q_1, \dots, q_k)$, denoted $\Delta_{\mathbb{C}}(q; m, k)$ if $q_j = q$ for all j , for which any m measures on \mathbb{C}^d can be equipartitioned by k complex regular q_j -fans?*

Other measure partitions arise by annihilating different Fourier coefficients. For instance, supposing that each $q_j = p_j r_j$, annihilating each c_i^ϵ in (3.6) other than those $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \bigoplus_{j=1}^k p_j \mathbb{Z}_{q_j}$ yields an equipartition of each μ_i “modulo” the subgroup

$$H = Z_{p_1} \times \dots \times Z_{p_k}:$$

$$(3.7) \quad \mu_i(\mathcal{R}_{hg}) = \mu_i(\mathcal{R}_g)$$

for each $g \in G$ and each $h \in H$. Equivalently, there is a collection F_{q_1}, \dots, F_{q_k} of complex regular q_j -fans, each of whose $r = \prod_{j=1}^k r_j$ sub-collections of regular p_j -fans F_{p_1}, \dots, F_{p_k} equipartition each measure. Such partitions are similar in flavor to those of Makeev [14] and then Blagojević and Karasev [2] on finding families of n orthogonal hyperplanes, any k of which equipartition a given set of measures:

Question 2. *Given any m measures on \mathbb{C}^d and a k -tuple (q_1, \dots, q_k) of integers, $q_j = p_j r_j$, does there exist a family of complex regular q_j -fans F_{q_1}, \dots, F_{q_k} , each of whose $\prod_{j=1}^k r_j$ sub-collections of complex regular p_j -fans F_{p_1}, \dots, F_{p_k} equipartition each measure?*

4. COMPLEX GRÜNBAUM PROBLEMS FOR REGULAR FANS

4.1. Lower Bounds. In the original Grünbaum problem, the lower bound $\Delta(m, k) \geq m(2^k - 1)/k$ (shown in [18], conjectured there to be optimal, and established as such in [15], [18], and [30] in a number of cases), follows by considering m disjoint segments on the moment curve $M = \{(t, t^2, \dots, t^d) \mid t \in \mathbb{R}\}$. For an equipartition by k hyperplanes, each segment must be intersected in $(2^k - 1)$ points, yielding $m(2^k - 1)$ points on M . However, the intersection of a hyperplane and M represents the root of a degree d polynomial, so $kd \geq m(2^k - 1)$.

A similar approach for general complex fans (i.e., the half-hyperplanes have arbitrarily prescribed dihedral angles) can be applied when $k = 1$ by considering points on the complex moment curve $M_{\mathbb{C}} = \{(z, z^2, \dots, z^d) \mid z \in \mathbb{C}\}$. In particular,

Proposition 4.1.

$$(4.1) \quad \Delta_{\mathbb{C}}(q; 1, m) \geq m \lfloor (q - 1)/2 \rfloor$$

if $q > 2$.

Proof. If any m (continuous) measures on \mathbb{C}^d can be equipartitioned by a complex q -fan with prescribed dihedral angles, then in particular the interior of the union of any two adjacent closed sectors contains at most $2/q$ of each total measure. Consider m collections C_1, \dots, C_m of $q' = \lfloor (q - 1)/2 \rfloor$ points lying on $M_{\mathbb{C}}$, and for each $\varepsilon > 0$ with the ε -balls centered at these points disjoint, let μ_i^ε be the union of the ε -balls with centers in C_i . A standard limiting argument (see, e.g., [26] or [29]) shows there exists a complex q -fan with the specified angles for which again the interior of the union of any two adjacent sectors contains at most $2/q$ points (hence none) of C_i . Each point of C_i must therefore lie on the fan itself, and since the interior of any of the fan's half-hyperplanes lies in the interior of the union of two adjacent sectors, each point of C_i lies on the complex hyperplane. As in the real case, the intersection of a point of $M_{\mathbb{C}}$ and a complex hyperplane is a root of a degree d polynomial with complex coefficients, so $d \geq q'm$. \square

Proposition 4.1 can be compared to the cases when the complex affine center requirement is dropped. For instance, $\Delta_{\mathbb{C}}(3; n, 1) = n$ (see [21] or Theorem 4.4 below), but it was shown in [27] that any $n - 1$ measures on \mathbb{R}^n can be equipartitioned by a single regular 3-fan.

Although we have not been able to prove it, we conjecture a lower bound for $\Delta_{\mathbb{C}}(m; q_1, \dots, q_k)$ when $k \geq 2$ similar to the one for $k = 1$ above:

Conjecture 1. *Let $Q = \prod_{j=1}^k q_j$, $q_j > 2$. Then*

$$k\Delta_{\mathbb{C}}(m; q_1, \dots, q_k) \geq m \lfloor (Q - 1)/2 \rfloor$$

4.2. Upper Bounds via G -Ham Sandwich Theorems. Now we give some upper bounds, obtained via Theorem 3.1. For the representation $\rho = m \oplus_{\epsilon \in [\Pi]} \chi^\epsilon$, $[\Pi] = \{\epsilon \neq 0 \mid \epsilon_k \leq (p - 1)/2\}$, the polynomial (3.4) is

$$(4.2) \quad f(y_1, \dots, y_k) = \prod_{\epsilon \in [\Pi]} (\epsilon_1 y_1 + \dots + \epsilon_k y_k)^m$$

As a simple application, one has

Proposition 4.2.

$$(4.3) \quad \Delta_{\mathbb{C}}(9; 1, 1) = 4$$

Proof. $f(y_1) = y_1 \cdot 2y_1 \cdot 3y_1 \cdot 4y_1 = 6y_1^4 \notin \langle y_1^5 \rangle \subset \mathbb{Z}_9[y_1]$. Thus $\Delta_{\mathbb{C}}(9; 1, 1) \leq 4$, so $\Delta_{\mathbb{C}}(9; 1, 1) = 4$ by Proposition 4.1. \square

The above proposition is to our knowledge the first equipartition to be obtained using (polynomial) cohomology rings with \mathbb{Z}_q -coefficients for non-prime q , despite the presence of zero divisors (the newer results $\Delta(2^{n+2} + 1, 2) = 3 \cdot 2^{n+1} + 2$ of [30] rest on an obstruction class representing 2 in an equivariant cohomology group isomorphic to \mathbb{Z}_4 , but do not involve the multiplicative structure of a cohomology ring).

For the elementary abelian groups Z_p^k , p an odd prime, one has $f(y_1, \dots, y_k) \in \mathbb{Z}_p[y_1, \dots, y_k]$, so that coefficients are taken in the field \mathbb{Z}_p usual. Resulting upper bounds on $\Delta_{\mathbb{C}}(p; m, k)$ then strongly parallel those of [15] on $\Delta(m, k)$ obtained from Z_2^k -cohomological index theory with \mathbb{Z}_2 -coefficients. As in Theorem 4.1 of [15], we consider the “Dickson” polynomial

$$(4.4) \quad D(p, k) = \text{Det} \begin{pmatrix} y_1 & \dots & y_k \\ y_1^p & \dots & y_k^p \\ \vdots & & \vdots \\ y_1^{p^{k-1}} & \dots & y_k^{p^{k-1}} \end{pmatrix} = \sum_{\sigma \in S_k} \text{sgn}(\sigma) y_{\sigma(1)}^p y_{\sigma(2)}^p \cdots y_{\sigma(k)}^p$$

Theorem 4.3. $\Delta_{\mathbb{C}}(p; m, k) \leq d$ if $D(p, k)^{m(p-1)/2} \notin \langle y_1^{d+1}, \dots, y_k^{d+1} \rangle$

Proof. It is shown in the proof of Proposition 1.1 of [28] that $D(p, k) \in \mathbb{Z}_p[y_1, \dots, y_k]$ is the product of all non-zero $\epsilon_{r,1}y_1 + \dots + \epsilon_{r,k}y_k$ whose last non-zero coordinate is 1. Thus $\prod_{\epsilon \in [\Pi]} (\epsilon_1 y_1 + \dots + \epsilon_k y_k)$ is a non-zero constant multiple of $D(p, k)^{(p-1)/2}$, $f(y_1, \dots, y_k)$ is a non-zero constant multiple of $D(p, k)^{m(p-1)/2}$, and so $f \neq 0$ iff $D(p, k)^{m(p-1)/2} \notin \mathcal{I} = \langle y_1^{d+1}, \dots, y_k^{d+1} \rangle$. \square

As $D(p, 1)^{m(p-1)/2} = y_1^{m(p-1)/2}$, one has $\Delta_{\mathbb{C}}(p; m, 1) \leq m(p-1)/2$ (see also [21]) and so by (4.1)

Corollary 4.4.

$$(4.5) \quad \Delta_{\mathbb{C}}(p; m, 1) = m(p-1)/2$$

for all odd primes p .

For $k \geq 2$, the sum of the exponents within each monomial of $D(p, k)^{m(p-1)/2}$ is $(1 + p + \dots + p^{k-1})m(p-1)/2 = m(p^k - 1)/2$, so the best possible upper bound via Theorem 4.3 is $\Delta_{\mathbb{C}}(p; m, k) \leq \lceil m(p^k - 1)/2k \rceil$, obtained when there is a monomial in $D(p, k)^{m(p-1)/2}$ of this degree. When k divides $m(p^k - 1)$, one then has the exact value $\Delta_{\mathbb{C}}(p; m, k) = m(p^k - 1)/2k$, provided the conjectured lower bound holds. For example, when $k = 2$ one has

Corollary 4.5. *Let p be an odd prime, and let $\sum_{i=1}^n a_i p^i$ be the base p expansion of $m(p-1)/2$, $0 \leq a_i < p$. If each a_i is even, then*

$$(4.6) \quad \Delta_{\mathbb{C}}(p; m, 2) \leq m(p^2 - 1)/4$$

Proof. Let $m' = m(p-1)/2$. The coefficient of the unique monomial $y_1^{m(p^2-1)/4} y_2^{m(p^2-1)/4}$ of degree $m(p^2 - 1)/4$ in $D(p, 2)^{m'} = (y_1 y_2^p - y_2 y_1^p)^{m'}$ is the binomial coefficient $\binom{m'}{m'/2}$, and $\binom{m'}{m'/2} = \prod_i \binom{a_i}{a_i/2} \neq 0$ by Lucas's theorem (see, e.g., [8]). \square

For instance, any two measures on \mathbb{C}^4 can be equipartitioned by a pair of complex regular 3-fans, and any measure on \mathbb{C}^6 can be equipartitioned by a pair of complex regular 5-fans. Letting each $a_i = p - 1$ yields $\Delta_{\mathbb{C}}(p; 2(p^{n+1} - 1)/(p - 1), 2) \leq \frac{p+1}{2} \cdot (p^{n+1} - 1)$, and in particular that $\Delta_{\mathbb{C}}(3; 3^{n+1} - 1, 2) \leq 2 \cdot (3^{n+1} - 1)$, which can be compared to the optimal inequality $\Delta(2^{n+1} - 1, 2) \leq \frac{3}{2} \cdot (2^{n+1} - 1)$ of [15]. On the other hand, the coefficient of $y_1^{m(p^2-1)/4} y_2^{m(p^2-1)/4}$ can vanish if there is a non-even coefficient in the base p expansion of $m(p-1)/2$, so the upper bounds given by Theorem 4.3 do not match the conjectured lower bound in these cases. For example, we can only show $2 \leq \Delta_{\mathbb{C}}(3; 1, 2) \leq 3$, where the lower bound is obtained by considering a single ball and the dimension of the intersection of two complex hyperplanes. The estimate $3 \leq \Delta_{\mathbb{C}}(3; 2, 2) \leq 4$ can be obtained analogously.

5. MAKEEV-TYPE PROBLEMS FOR COMPLEX REGULAR FANS

When $G \neq Z_p^k$, p an odd prime, there are clear limitations on using the representation $\rho = m \oplus_{\epsilon \in [\Pi]} \chi^\epsilon$, i.e., of simultaneously eliminating half off all the non-zero Fourier coefficients. For instance, if $q = pr$ is a product of distinct odd primes, one cannot use ρ to show that $\Delta_{\mathbb{C}}(q; m, 1) = (q-1)m/2$, since $py_1 \cdot ry_1 = 0$ in \mathbb{Z}_q and so $f(y_1) = 0$. It is precisely for this problem of zero divisors that general \mathbb{Z}_q -coefficients are avoided in topological combinatorics. Nonetheless, one does obtain some answers to Question 2 in such cases.

When $k = 1$ and $q = pr$, p an odd prime, the $f(y_1)$ of (3.4) arising from annihilating each c_i^ϵ for which $\epsilon \leq \lceil (q-1)/2 \rceil$ and $p \nmid \epsilon$ is $f(y_1) = (\Pi_\epsilon) y_1^d \in \mathbb{Z}_q[y_1]$, $d = mr(p-1)/2$, which is not in $\langle y_1^{d+1} \rangle$ since $\Pi_\epsilon \epsilon$ is not divisible by p . Therefore

Proposition 5.1. *Let $q = pr$, where p is an odd prime. For any m measures on $\mathbb{C}^{mr(p-1)/2}$, there exists a complex regular q -fan, each of whose r complex regular p -fans equipartitions each measure.*

Thus even though $\Delta_{\mathbb{C}}(15; m, 1) \geq 7m$, given any m measures on \mathbb{C}^{5m} there still exists a regular 15-fan, each of whose 5 regular 3-fans equipartitions each measure, and likewise for m measures on \mathbb{C}^{6m} , one whose 3 regular 5-fans equipartitions.

For $k > 1$, let p again be an odd prime and consider $G = \Pi_{j=1} Z_{q_j}$, $q_j = pr_j$. As in section 4, we consider the Dickson polynomial $D(p, k) \in \mathbb{Z}_p[y_1, \dots, y_k]$.

Theorem 5.2. *Let $q_1 = pr_1, \dots, q_k = pr_k$, p an odd prime, and let $r = \Pi_{j=1}^k r_j$. If $D(p, k)^{rm(p-1)/2} \notin \langle y_1^{d+1}, \dots, y_k^{d+1} \rangle$, then for any m measures on \mathbb{C}^d there exists $F_{q_1}, F_{q_2}, \dots, F_{q_k}$, each of whose r sub-collections of k complex regular p -fans equipartitions each measure.*

Proof. Let $[\Pi] = \{\epsilon \neq 0 \mid \epsilon_k \leq \lceil (q_k - 1)/2 \rceil\}$ as before and let $[\Pi]_r$ be the subset which excludes all $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ for which $p \mid \epsilon_j$. One then has $f(y_1, \dots, y_k) = g(y_1, \dots, y_k)^m$, where $g(y_1, \dots, y_k) = \Pi_{\epsilon \in [\Pi]_r} (\epsilon_1 y_1 + \dots + \epsilon_k y_k)$. Reducing mod p , $g = h^r$, where $h = \Pi_{\delta} (\delta_1 y_1 + \dots + \delta_k y_k) \in \mathbb{Z}_p[y_1, \dots, y_k]$, the product taken over all $\delta \in \mathbb{Z}_p^k - \{0\}$ with $\delta_k \leq (p-1)/2$. As in the proof of Theorem 4.3, h is a non-constant multiple of $D(p, k)^{(p-1)/2}$, so f is a non-constant multiple of $D(p, k)^{rm(p-1)/2}$ when reduced mod p , and therefore $f \neq 0$ if $D(p, k)^{mr(p-1)/2} \notin \langle y_1^{d+1}, \dots, y_k^{d+1} \rangle$. \square

When $k = 2$, optimal dimensions in Theorem 5.2 arise under the same circumstances as in the non-modulo complex Grünbaum problem:

Corollary 5.3. *Let $mr(p-1)/2 = \sum_{i=1}^n a_i p^i$ with each a_i is even. Then for any m measures on \mathbb{C}^d , $d = mr(p^2 - 1)/4$, there exists complex regular fans F_{pr_1} and F_{pr_2} , each of whose $r = r_1 r_2$ sub-pairs of complex regular p -fans equipartitions each measure.*

For example, given two measures on \mathbb{C}^{16} , there exists a pair of complex regular 6-fans, each of whose four pairs of complex regular 3-fans equipartitions each measure, and for any single measure on \mathbb{C}^{24} , there exists a pair of complex regular 10-fans, each of whose four pairs of complex regular 5-fans equipartitions the measure.

5.1. Towards Equipartitions by Arbitrary Fans. The “modulo” partitions above do not by themselves yield equipartitions by arbitrary collections complex regular fans. However, we believe the later should follow by successively annihilating coefficients c_i^ϵ corresponding to a carefully chosen partition C_1, \dots, C_t of $\oplus_{j=1}^k \mathbb{Z}_{q_j} - \{0\}$. For each C_s , one has has a parametrized family M_s of G -partitions, and hence an equipartition of measures if the intersection of the families M_1, \dots, M_t is non-empty. In this way, one would be able to overcome the vanishing of $f(y_1, \dots, y_k)$ for $\rho = m \oplus_{\epsilon \in [\Pi]} \chi^\epsilon$ when G is not elementary abelian.

We sketch the idea for complex regular fans F_p and F_r when p and r are distinct odd primes. Let C_1 consist of those ϵ with $\epsilon \notin 0 \times \mathbb{Z}_r$ and C_2 contain all non-zero $\epsilon \in 0 \times \mathbb{Z}_r$. One can annihilate each c_i^ϵ for $\epsilon \in C_1$ if $d \geq mr(p-1)/2$: as $y_1 y_2 = 0$, $f(y_1, y_2) = \prod_{\substack{1 \leq \epsilon_1 \leq (p-1)/2 \\ 0 \leq \epsilon_2 < q}} (\epsilon_1 y_1 + \epsilon_2 y_2) = \kappa y_1^{mr(p-1)/2} \notin \langle py_1, ry_2, y_1^{d+1}, y_2^{d+1} \rangle$, and one can also find a G -partition with $c_i^\epsilon = 0$ for each $\epsilon \in C_2$ provided $d \geq m(r-1)/2$, since here $f(y_1, y_2) = (\prod_{\epsilon_2=1}^{r-1} \epsilon_2) y_2^{m(r-1)/2}$.

As the proof of Theorem 3.1 shows, a $G = (Z_p \times Z_r) \cong Z_{pr}$ -partition which annihilates the coefficients arising from C_1 represents an element of the pull-back $M_1 = h_1^{-1}(0)$ of the G -equivariant map $h_1 : N \rightarrow \mathbb{C}^{mr(p-1)/2}$, $N = [S(\mathbb{C}^{d+1}) - (0 \times S^1)]^2$, and likewise a G -partition annihilating the coefficients arising from C_2 is an element of $M_2 = h_2^{-1}(0)$ under a G -equivariant map $h_2 : N \rightarrow \mathbb{C}^{m(r-1)/2}$. Thus $\Delta_{\mathbb{C}}(m; p, r) \leq m(pr-1)/4$, provided one can show that $M_1 \cap M_2 \neq \emptyset$ when $d \geq m(pr-1)/4$. The plausibility of such a non-empty intersection can be seen by restricting to (positive) absolutely continuous measures $\mu = f dm$ with $f \in C_c^\infty(\mathbb{C}^d)$, as it is a standard fact that for the usual class of “admissible” measures (see, e.g., [26]), i.e., those which are the “weak limits” of absolutely continuous measures, the desired equipartitions in \mathbb{C}^d follow once $M_1 \cap M_2 \neq \emptyset$ is shown to be true for the more restrictive $\mu = f dm$. In these cases, the maps h_1 and h_2 are smooth and Proposition 6.1 assures that zero is a value for both maps, hence is a regular value in the generic cases. Thus M_1 and M_2 are each generically smooth (and compact) free G -submanifolds of N of real codimension $mr(p-1)$ and $m(r-1)$, respectively. By dimension considerations, M_1 and M_2 are G -transverse, so that a non-trivial intersection seems likely.

6. PROOF OF THEOREM 3.1

Our proof of Theorem 3.1 is a standard example of the “configuration-space/test-map paradigm”, the established method by which nearly all problems in geometric and discrete combinatorics are solved by topological means.

Proof. As discussed in section 3, $E_d(G) = [S(\mathbb{C}^{d+1})]^k$ realizes all the regions of all possible G -partitions of \mathbb{C}^d . In order to ensure continuity of our construction, however, we take for our “configuration-space” \tilde{X} what remains of $E_d(G)$ when a copy of Z_{q_j} is removed from each $S(\mathbb{C}^{d+1})$, where each $Z_{q_j} \hookrightarrow 0 \times S^1$ is embedded as the last coordinate of $S(\mathbb{C}^{d+1})$: $\tilde{X} = \prod_{j=1}^k \tilde{X}_j$, $\tilde{X}_j = S(\mathbb{C}^{d+1}) - Z_{q_j}$. All the non-trivial G -partitions of \mathbb{C}^d are still realized by the G -orbits of \tilde{X} , with a formula for the regions of each G -partition of \mathbb{C}^d given explicitly by

$$(6.1) \quad \mathcal{R}_\zeta(x) = \{\mathbf{u} \in \mathbb{C}^d \mid (\forall 1 \leq j \leq k) (\exists v_j \in R_{r_j}(q_j)) \langle \mathbf{u}, \mathbf{a}_j \rangle_{\mathbb{C}} + \bar{b}_j = v_j\},$$

$x = (\mathbf{a}_1, b_1, \dots, \mathbf{a}_k, b_k) \in \tilde{X}$, and we still have that $\mathcal{R}_{\zeta_1}(\zeta_2 \cdot x) = \mathcal{R}_{\zeta_1 \zeta_2}(x)$ for each $\zeta_1, \zeta_2 \in G$.

Our “target space” is \mathbb{C}^n , the representation space for $\rho : G \rightarrow U(n)$. Defining $h(x) = \sum_{\zeta \in G} \rho_\zeta^{-1}(\mu(\mathcal{R}_\zeta(x)))$ for our given n -tuple of complex-valued measures defines a “test-map” $h : \tilde{X} \rightarrow \mathbb{C}^n$. The exclusion of each Z_{q_j} assures that each $F_{q_j}(x) = \{\mathbf{u} \in \mathbb{C}^d \mid (\exists 0 \leq r < q_j) \langle \mathbf{u}, \mathbf{a}_j \rangle_{\mathbb{C}} + \bar{b}_j = \zeta_{q_j}^r\}$, is either a complex regular q_j -fan (if

$\mathbf{a}_j \neq 0$) or the emptyset (if $\mathbf{a}_j = 0$), hence is always a null-set, and a nearly identical argument to the one given in [21] (the case $k = 1$) shows that this map is continuous. Crucially, this map is also G -equivariant – $h(\zeta \cdot x) = \rho_\zeta(h(x))$ for each $x \in \tilde{X}$ and $\zeta \in G$. Theorem 3.1 therefore follows once it is shown that 0 lies in the image of our test map, and moreover that any $x \in \tilde{X}$ for which $h(x) = 0$ determines a non-trivial G -partition. The former is demonstrated by the Borsuk-Ulam type proposition below. The latter follows from the observation that otherwise $x \in (0 \times S^1)^k$, $x_j \notin Z_{q_j}$, so that by (6.1) $\mathcal{R}_\zeta(x) = \emptyset$ for all but one of the $\mathcal{R}_\zeta(x)$ and that this $\mathcal{R}_\zeta(x)$ must be of all of \mathbb{C}^d . The G -balancing equation would then yield $\rho_{\zeta^{-1}}(\mu(\mathbb{C}^d)) = 0$ and hence that $\mu(\mathbb{C}^d) = 0$. \square

Proposition 6.1. *Let $\rho = \oplus_{r=1}^n \chi^{\epsilon_r} : G \rightarrow U(n)$ be a unitary representation, $\epsilon_r \in \oplus_{j=1}^k \mathbb{Z}_{q_j}$, and suppose that $f(y_1, \dots, y_k) = \prod_{r=1}^n (\epsilon_{r,1}y_1 + \dots \epsilon_{r,n}y_k) \in \mathbb{Z}[y_1, \dots, y_k]/\mathcal{I}$ is non-zero, $\mathcal{I} = \langle q_j y_j, y_j^{d+1} \mid 1 \leq j \leq k \rangle$. Then for any continuous G -equivariant map $h : \tilde{X} \rightarrow \mathbb{C}^n$, there exists some $x \in \tilde{X}$ with $h(x) = 0$.*

Proof. The proposition can be reduced to calculating the top Chern class of a section of a complex vector bundle $E(\xi)$ over the quotient space $X := \tilde{X}/G$. Namely, the standard G -action on \tilde{X} and the G -action on \mathbb{C}^n from $\rho : G \rightarrow U(n)$ give rise to $E(\xi) = \tilde{X} \times_G \mathbb{C}^n$, the quotient of the trivial bundle $\tilde{X} \times \mathbb{C}^n$ under the diagonal G -action. The G -equivariant section $\tilde{s} : \tilde{X} \rightarrow \tilde{X} \times \mathbb{C}^n$ of the trivial bundle, $\tilde{s}(x) = (x, h(x))$, induces a section $s : X \rightarrow E(\xi)$, and h maps to zero iff this section vanishes. It is a standard fact (see, e.g., [17]) that the existence of a nowhere vanishing section is precluded by the non-vanishing of the top Chern class $c_n(\xi) \in H^{2n}(X; \mathbb{Z})$, which we demonstrate by identifying $\mathbb{Z}[y_1, \dots, y_k]/\mathcal{I}$ as a subring of $H^*(X; \mathbb{Z})$ and $c_n(\xi)$ with the non-zero polynomial $f(y_1, \dots, y_k)$.

The base space $X = \prod_{j=1}^k X_j$ is the product of the punctured Lens spaces $X_j = \tilde{X}_j/Z_{q_j} = L^{2d+1}(q_j) - pt$, and each X_j deformation retracts onto the $2d$ -skeleton of the infinite dimensional Lens space $L^\infty(q_j)$, the classifying space $BZ_{q_j} = S(\mathbb{C}^\infty)/Z_{q_j} = EZ_{q_j}/Z_{q_j}$ of the universal bundle. In particular, X is homotopy equivalent to a sub-complex of the $2dk$ -skeleton of $BG = \prod_{j=1}^k L^\infty(q_j) = (S(\mathbb{C}^\infty))^k/G = EG/G$.

It will not be necessary to calculate the entire ring structure of $H^*(X)$, only its “tensor subring”, i.e., the image of the injection $\otimes_{j=1}^k p_j^* : \otimes_{j=1}^k H^*(X_j) \rightarrow H^*(X)$ induced from the coordinate projections $p_j : X \rightarrow X_j$ (this is an injection by the general Künneth formula (see, e.g., [11])). The calculation of this tensor subring is basic (essentially a chase of the diagram below), obtained by first calculating the tensor subring of $H^*(BG)$, which can itself be expressed entirely in terms of the first Chern classes of the k -fold product of complex projective space $(\mathbb{C}P^\infty)^k = (S(\mathbb{C}^\infty))^k/T^k = ET^k/T^k$, the classifying space of the Torus group $T^k = (S^1)^k$. This yields a more elementary calculation than if the cohomological index theory of [7] typically used in such circumstance were applied, in which case one would in particular need the considerably more complicated full ring structure of $H^*(BG; \mathbb{Z})$ (see, e.g., [6]).

It is a very classical fact that $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[c_1(\gamma)]$, where $c_1(\gamma) \in H^2(\mathbb{C}P^\infty)$ is the first Chern class of the canonical line bundle $E(\gamma) = S(\mathbb{C})^\infty \times_{S^1} \mathbb{C} \rightarrow \mathbb{C}P^\infty$,

the quotient being taken with respect to the standard diagonal action (see, e.g., [12] or [17]). Thus $H^*(BT^k) = \mathbb{Z}[c_1(\gamma_1), \dots, c_1(\gamma_k)]$ by the Künneth Formula and the naturality of Chern classes, where $E(\gamma_j) = (S(\mathbb{C}^\infty)^\infty \times_{T^k} \mathbb{C} \rightarrow BT^k$ is the pull-back bundle $p_j^*(E(\gamma))$ of $E(\gamma)$ under the projection $p_j : BT^k \rightarrow \mathbb{C}P^\infty$.

It is a standard fact that $H^*(L^\infty(q_j)) = \mathbb{Z}[u_j]/\langle q_j u_j \rangle$ (see, e.g., [11]). Considering the standard CW structures as in [11], each projection $\pi_j : L^\infty(q_j) \rightarrow L^\infty(q_j)/S^1 = \mathbb{C}P^\infty$ sends each even-dimensional cell of $L^\infty(q_j)$ homeomorphically onto each even-dimensional cell of $\mathbb{C}P^\infty$, so that $\pi_j^* : H^*(\mathbb{C}P^\infty; \mathbb{Z}) \rightarrow H^*(L^\infty(q_j); \mathbb{Z})$ represents the projection $\mathbb{Z} \rightarrow \mathbb{Z}_m$ in positive even dimensions. In particular, the generator $u_j \in H^2(L^\infty(q_j))$ may be taken to be $u_j = \pi_j^*(c_1(\gamma))$. The image of $\pi^* : H^*(BT^k) \rightarrow H^*(BG)$ under the projection $\pi = \prod_{j=1}^k \pi_j : BG \rightarrow BT^k$ is the tensor subring of $H^*(BG)$, which is therefore $\mathbb{Z}[b_1, \dots, b_k]/\langle q_1 b_1, \dots, q_k b_k \rangle$, where $b_j = p_j^*(u_j)$ and $p_j : BG \rightarrow L^\infty(q_j)$ is the projection. Letting $E(\gamma(q_j)) = (S(\mathbb{C}^\infty))^k \times_G \mathbb{C} \rightarrow X$ be the complex line bundle associated to the “basis” representation $\chi^{e_j} : G \rightarrow U(1)$, $e_j = (0, \dots, 1, \dots, 0) \in \oplus_{j=1}^k \mathbb{Z}_{q_j}$, it is easily seen that each $E(\gamma(q_j)) = \pi^*(E(\gamma_j))$ and hence that $b_j = c_1(\gamma(q_j))$.

An easy cellular cohomology argument shows that $H^*(X_j) = \mathbb{Z}[v_j]/\langle q_j v_j, v_j^{d+1} \rangle$ is the image of the restriction $i_j^* : H^*(L^\infty(q_j)) \rightarrow H^*(X_j)$, $v_j = i_j^*(u_j)$, and therefore the image of the tensor subring of $H^*(BG)$ under $i^* = (i_1 \times \dots \times i_k)^* : H^*(BG) \rightarrow H^*(X)$ is the tensor subring of $H^*(X)$:

$$\mathbb{Z}[y_1, \dots, y_k]/\mathcal{I}, \quad \mathcal{I} = \langle q_1 y_1, y_1^{d+1}, \dots, q_k y_k, y_k^{d+1} \rangle,$$

where $y_j = i^*(b_j)$. Letting $E(\xi_j) = \tilde{X} \times_G \mathbb{C}$ be the line bundles associated to χ^{e_j} , it is immediate that $E(\xi_j) = i^*(E(\gamma(q_j)))$ and hence that $y_j = c_1(\xi_j)$. This is summarized by the following diagram:

$$\begin{array}{ccccc} E(\xi_j) & \longrightarrow & E(\gamma(q_j)) & \longrightarrow & E(\gamma_j) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i} & BG & \xrightarrow{\pi} & BT^K \\ p_j \downarrow & & p_j \downarrow & & p_j \downarrow \\ X_j & \xrightarrow{i_j} & L^\infty(q_j) & \xrightarrow{\pi_j} & \mathbb{C}P^\infty \end{array}$$

Finally, we show that $f(y_1, \dots, y_k) \in \mathbb{Z}[y_1, \dots, y_k]/\mathcal{I}$ is just $c_n(\xi)$. This follows from the well-known fact (see, e.g., [12]) that the non-isomorphic complex line bundles over X , $Vect_{\mathbb{C}}^1(X)$, form a group under tensor products and that the map $c_1 : Vect_{\mathbb{C}}^1(X) \rightarrow H^2(X; \mathbb{Z})$ sending each bundle to its first Chern class is a group isomorphism. As $E(\xi) = \oplus_{r=1}^n E(\xi^r)$ is the Whitney sum of the tensors $E(\xi^r) := \otimes_{j=1}^k E(\xi_j)^{\otimes \epsilon_{r,j}}$, the Whitney sum formula [17] shows that the total Chern class $c(\xi)$ is the product $c(\xi) = c(\xi^1) \cdots c(\xi^k) = (1 + c_1(\xi^1)) \cdots (1 + c_1(\xi^k))$ and hence that $c_n(\xi) = c_1(\xi^1) \cdots c_1(\xi^k)$. Since $Vect_{\mathbb{C}}^1(X) \cong H^2(X; \mathbb{Z})$, one has $c_1(\xi^r) = \epsilon_{r,1} c_1(\xi_1) + \dots + \epsilon_{r,k} c_1(\xi_k)$ for each $1 \leq r \leq n$, so that $c_n(\xi) = \prod_{r=1}^n (\epsilon_{r,1} y_1 + \dots + \epsilon_{r,k} y_k) = f(y_1, \dots, y_k)$, as desired. \square

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